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Best Approximation of Lebesgue–Bochner Summable Functions

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In this paper the approximation properties of the space of functions with values in a Banach space which are summable in the sense of Bochner [1] are studied. We obtain some theorems on the characterization and unicity of best approximations related to those given by Cheney and Wulbert [5], Kripke and Rivlin [10], Pták [11], and Singer [12] for real-valued summable functions. In Section 4 we apply our results to an approximation problem similar to that investigated by Carroll and McLaughlin [4].

1. DEFINITIONS

Let I be the interval $[0, 1]$, μ the Lebesgue measure on I, and X a real Banach space with the norm $|\cdot|$. The class of all summable functions $x(t)$ on I into X (for the definition of measurability and summability see [1]) provided with the norm

$$\| x \| = \int_I | x(t)| \, dt$$

is a Banach space and will be denoted by $L(X)$. Following Bochner and Taylor [3] we denote further by $V^\infty(X)$ the class of all functions $x(t)$ on I into X such that $x(0) = 0$ and such that there is a constant $A \geqslant 0$ with the property $| x(t + h) - x(t)| \leqslant A \, | h |$, whenever t and $t + h$ belong to I. Denoting by $N(x)$ the least such constant and defining the norm on $V^\infty(X)$ by $\| x \| = N(x)$, $V^\infty(X)$ is a Banach space. The space of all continuous functions $x(t)$ on I into X provided with the norm $\| x \| = \sup_I | x(t)|$ will be denoted by $C(X)$. Let X^* be the conjugate space of X. For $x \in L(X)$ and $u \in V^\infty(X^*)$ let the integral

$$\int_I du(t) \, x(t)$$

be defined in the same way as in [3].

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The following theorem gives the general form of linear bounded functionals on $L(X)$.

THEOREM 1.1 (Bochner and Taylor [3]). *For every $f \in L^*(X)$ there is a $u \in V^\infty(X^*)$ such that $\| u \| = \| f \|$ and*

$$f(x) = \int_I du(t)\, x(t) \tag{1.1}$$

for every $x \in L(X)$. Conversely, for every $u \in V^\infty(X^*)$, (1.1) *defines a linear bounded functional f on $L(X)$ with $\| f \| = \| u \|$.*

Let X be a normed linear space, E a subspace of X. For $x \in X$ we denote by $P_E(x)$ the set of all best approximations of x in E, i.e., $P_E(x) = \{e_0 \in E; \| x - e_0 \| = \inf_{e \in E} \| x - e \|\}$. E is said to be a U-space if for every $x \in X$ the set $P_E(x)$ contains at most one element of E. We further denote $E^0 = \{x \in X; 0 \in P_E(x)\}$ and for a function x on I to X, $R(x) = \{t; x(t) \neq 0\}$ and $Z(x) = \{t; x(t) = 0\}$. $\mathbb{R}$ and $\mathbb{N}$ will denote the set of all real numbers and the set of all positive integers, respectively.

2. Maximal Functionals on $L(X)$

In the main theorem of this section we give characterizations in terms of "differentiability" of those functionals u on $L(X)$ which are maximal, i.e., for which there is a $x \in L(X)$ with $ux = \| u \| \cdot \| x \|$. In the particular case that every function u on I to X which has bounded variation possesses the strong derivative a.e. in I, this theorem is a consequence of Theorem 2.2 [3]. There are, however, as shown in [2], a Banach space X and a function $u \in V^\infty(X)$ such that u is not differentiable at any point in I.

We denote

$$u'(t)\, x(t) = \lim_{h \to 0} \frac{u(t + h) - u(t)}{h}\, x(t)$$

for $u \in V^\infty(X^*)$, $x \in L(X)$, and every $t \in I$ for which the limit exists.

LEMMA 2.1. *Let $Q \subset I$ be a closed set, X a Banach space, $x\colon Q \to X$ a continuous function, $u \in V^\infty(X^*)$, $\| u \| = 1$. Let*

$$\int_Q du(t)\, x(t) = \int_Q | x(t) |\, dt. \tag{2.1}$$

Then we have

$$u'(t)\, x(t) = | x(t) | \qquad a.e.\ in\ Q.$$

Proof. Since $\| u \| = 1$, (2.1) implies

$$\int_A du(t)\, x(t) = \int_A | x(t)|\, dt \tag{2.2}$$

for every measurable subset A of Q. The map x being continuous and bounded by a constant $M \geqslant 1$, it may be extended to a continuous map on the whole interval I bounded by the same constant M ([6, Theorem IX, 6.1]). This extension will be denoted by x again. For $t \in Q$, $h \neq 0$, $z: Q \to X$ let us denote

$$D(t, h, z) = ((u(t + h) - u(t))/h)\, z(t).$$

The function $\liminf_{h\to 0} D(t, h, x)$ is measurable (this fact may be proved in a way similar to that used to prove the measurability of the lower derivative of a real function). Since $\| u \| = 1$, we have for every $t \in Q$

$$\limsup_{h\to 0} D(t, h, x) \leqslant | x(t)|.$$

We denote

$$G = \{t \in Q; \liminf_{h\to 0} D(t, h, x) < | x(t)|\},$$

$$G_n = \{t \in Q; \liminf_{h\to 0} D(t, h, x) < | x(t)| - 1/n\}, \qquad n \in \mathbb{N}.$$

Obviously $G = \bigcup_{n=1}^{\infty} G_n$ and the proof of the lemma will be completed by showing that for every $n \in \mathbb{N}$, $\mu(G_n) = 0$. Assume the converse. Then there is a $n_0 \in \mathbb{N}$ such that for $G_0 = G_{n_0}$ we have $\mu(G_0) = a > 0$.

Let ϵ be an arbitrary real number, $0 < \epsilon < a/2$. Then there exist an open set H, $G_0 \subset H$, a closed set F, $F \subset G_0$ and a continuous real function $f: I \to I$ with $f(t) = 1$ for every $t \in F$, $f(t) = 0$ for every $t \in I \backslash H$ such that

$$\mu(H \backslash G_0) < \epsilon/4M, \qquad \mu(G_0 \backslash F) < \epsilon/4M, \tag{2.3}$$

$$\left| \int_{G_0} | x(t)|\, dt - \int_I | y(t)|\, dt \right| < \epsilon, \tag{2.4}$$

and

$$\left| \int_{G_0} du(t)\, x(t) - \int_I du(t)\, y(t) \right| < \epsilon, \tag{2.5}$$

where $y = f \cdot x$. Since y is a continuous function, there is a $\delta > 0$ such that for an arbitrary partition P: $t_0, \ldots, t_n$ of I and arbitrary points $\tau_i \in [t_{i-1}, t_i]$, $i = 1, \ldots, n$, $| P | = \max_i | t_i - t_{i-1} | < \delta$ implies

$$\left| \int_I du(t)\, y(t) - S(P) \right| < \epsilon, \tag{2.6}$$

where $S(P) = \sum_{j=1}^{n} (u(t_j) - u(t_{j-1}))\, y(\tau_j)$ and

$$\left| \int_I |\, y(t)|\, dt - \sum_{j=1}^{n} |\, y(\tau_j)|\, (t_j - t_{j-1}) \right| < \epsilon. \tag{2.7}$$

Since $F \subset G_0$, there exists for every $t \in F$ a sequence $\{h_k(t)\}_{k\in\mathbb{N}}$ with the properties

$$0 < |\, h_k(t)| < \delta \qquad \text{for every } k \in \mathbb{N} \text{ and every } t \in F,$$

$$\lim_{k\to\infty} h_k(t) = 0 \qquad \text{for every } t \in F,$$

and

$$D(t, h_k(t), y) < |\, y(t)| - 1/n_0 \tag{2.8}$$

for every $t \in F$ and $k \in \mathbb{N}$. The class of all intervals $[t, t + h_k(t)]$ (or $[t + h_k(t), t]$ if $h_k(t) < 0$), $t \in F$, covers the set F in the sense of Vitali [7]. Hence it has a finite disjoint subclass $I(t_i)$, $i = 1,\dots, m$, such that

$$\mu\left(F \Big\backslash \bigcup_{i=1}^{m} I(t_i)\right) < \epsilon/2M. \tag{2.9}$$

Let P_0 be a partition of I consisting of all boundary points t_i, $t_i + h_i$ of the intervals $I(t_i)$, $i = 1,\dots, m$. Let $P_1 : s_0 ,\dots, s_p$ be a refinement of P_0 such that none of the points s_j, $j = 0,\dots, p$, is contained in any open interval Int $I(t_i)$, $i = 1,\dots, m$.

For every index $j = 1,\dots, p$ exactly one of the following conditions can be satisfied:

(i) $[s_{j-1}, s_j] = I(t_i)$ for some $i = 1,\dots, m$. In this case we put $\tau_j = t_i$.

(ii) $[s_{j-1}, s_j] \cap (I \backslash H) \neq \varnothing$ and (i) does not hold. In this case we choose an arbitrary $\tau_j \in [s_{j-1}, s_j] \cap (I \backslash H)$.

(iii) $[s_{j-1}, s_j] \cap (I \backslash H) = \varnothing$ and (i) does not hold. In this case we choose an arbitrary $\tau_j \in [s_{j-1}, s_j]$.

Denoting by B the set of all indices j which satisfy condition (iii), we have by (2.3) and (2.8)

$$\mu\left(\bigcup_{j\in B} [s_{j-1}, s_j]\right) \leqslant \mu\left(H \Big\backslash \bigcup_{i=1}^{m} I(t_i)\right)$$

$$\leqslant \mu(H \backslash F) + \mu\left(F \Big\backslash \bigcup_{i=1}^{m} I(t_i)\right) < \epsilon/M.$$

Hence, we have by (2.4)–(2.9)

$$\int_{G_0} du(t)\,x(t) \leqslant \int_I du(t)\,y(t) + \epsilon < \sum_{j=1}^{p} (u(s_j) - u(s_{j-1}))\,y(\tau_j) + 2\epsilon$$

$$\leqslant \sum_{i=1}^{m} D(t_i, h_i, y)\,|\,h_i\,| + \sum_{j\in B} (u(s_j) - u(s_{j-1}))\,y(\tau_j) + 2\epsilon$$

$$\leqslant \sum_{i=1}^{m} (|\,y(t_i)| - 1/n_0)\,|\,h_i\,| + M\cdot\mu\left(\bigcup_{j\in B}[s_{j-1}, s_j]\right) + 2\epsilon$$

$$\leqslant \sum_{i=1}^{m} |\,y(t_i)|\,|\,h_i\,| - \sum_{i=1}^{m} (1/n_0)\,|\,h_i\,| + 3\epsilon$$

$$\leqslant \sum_{j=1}^{p} |\,y(\tau_i)|\,(s_j - s_{j-1}) - \sum_{i=1}^{m} (1/n_0)\,|\,h_i\,| + 3\epsilon.$$

Since by (2.3) and (2.9), $\sum_{i=1}^{m} |\,h_i\,| = \mu(\bigcup_{i=1}^{m} I(t_i)) > a/2$, we have by (2.4) and (2.7) for every ϵ, $0 < \epsilon < a/2$,

$$\int_{G_0} du(t)\,x(t) < \int_{G_0} |\,x(t)|\,dt - a/2n_0 + 5\epsilon.$$

This, however, for $\epsilon < a/10n_0$, contradicts (2.2).

LEMMA 2.2. *For $u \in V^\infty(X^*)$, $x \in L(X)$, let $u'(t)\,x(t)$ exist a.e. in I. Then*

$$\int_I u'(t)\,x(t)\,dt = \int_I du(t)\,x(t). \tag{2.10}$$

Proof. Since

$$|\,u'(t)\,x(t)| = \lim_{h\to 0} 1/|\,h\,|\,|(u(t+h) - u(t))\,x(t)|$$
$$\leqslant \|\,u\,\|\,|\,x(t)|\ \text{a.e. in } I, \tag{2.11}$$

$u'(t)\,x(t)$ is summable. Let $\epsilon > 0$ be given. Then there is a $\delta > 0$ such that for every measurable set $G \subset I$, $\mu(G) < \delta$ implies

$$\int_G |\,x(t)|\,dt < \epsilon/\|\,u\,\|. \tag{2.12}$$

By Lusin's theorem there is a $y \in C(X)$ such that $\mu(R(x - y)) < \delta/3$ and

$$\int_I |\,x(t) - y(t)|\,dt < \epsilon/\|\,u\,\|, \tag{2.13}$$

and a $v \in C(\mathbb{R})$ such that $\mu(R(v - u'x)) < \delta/3$ and

$$\int_I |u'(t)\,x(t) - v(t)|\,dt < \epsilon. \tag{2.14}$$

We have $v(t) = u'(t)\,x(t)$ for every $t \in N = (I \backslash (R(x - y) \cup R(v - u'x))) \backslash M$, $\mu(M) = 0$. We choose an η, $0 < \eta < \delta$, such that for every partition P, $|P| < \eta$ implies

$$\left| S(P) - \int_I du(t)\,y(t) \right| < \epsilon \tag{2.15}$$

and $|t_1 - t_2| < \eta$ implies

$$|v(t_1) - v(t_2)| < \epsilon. \tag{2.16}$$

For every $t \in N$ there exists a sequence $h_n(t) \to 0$ such that for every $n \in \mathbb{N}$, $0 < h_n(t) < \eta$ and

$$|D(t, h_n(t), x) - u'(t)\,x(t)| < \epsilon. \tag{2.17}$$

Since the class of all intervals $[t, t + h_n(t)]$, $t \in N$, $n \in \mathbb{N}$, covers the set N in the sense of Vitali, there is a $k \in \mathbb{N}$ and a disjoint subclass $[t_j, t_j + h_j]$, $j = 1,..., k$, such that we have

$$\mu\left(N \backslash \bigcup_{j=1}^{k} [t_j, t_j + h_j]\right) < \delta/3. \tag{2.18}$$

Let P: $s_0,..., s_n$ be a refinement of the partition P_1: $t_1,..., t_n + h_n$ such that $|P| < \eta$, there is no index $i = 0,..., n$ such that $s_i \in \operatorname{Int}[t_j, t_j + h_j]$ for some $j = 1,..., k$ and

$$\left| \int_{I \backslash \bigcup_{j=1}^{k}[t_j, t_j+h_j]} du(t)\,y(t) - \sum_{\substack{i=0 \\ i \notin A}}^{n} (u(s_{i+1}) - u(s_i))\,y(s_i) \right| < \epsilon, \tag{2.19}$$

where A is the set of all indices i such that there is a $j = 1,..., k$ with $s_i = t_j$. Let us put

$$w(t) = \sum_{i=0}^{n} \frac{u(s_{i+1}) - u(s_i)}{s_{i+1} - s_i}\, y(s_i)\, \chi_{[s_i, s_{i+1}]}(t),$$

where χ is the characteristic function. By (2.15), we have

$$\left| \int_I w(t)\,dt - \int_I du(t)\,y(t) \right| < \epsilon.$$

Further, by (2.11)–(2.14), (2.16), (2.17), (2.19), we have

$$\left| \int_I w(t)\,dt - \int_I v(t)\,dt \right|$$

$$\leqslant \sum_{j=1}^{k} \int_{[t_j, t_j+h_j]} | D(t_j, h_j, x) - v(t)|\,dt$$

$$+ \int_I \left| \sum_{\substack{i=0 \\ i \notin A}}^{n} \left(\frac{u(s_{i+1}) - u(s_i)}{s_{i+1} - s_i} y(s_i) - v(t) \right) \chi_{[s_i, s_{i+1}]}(t) \right| dt$$

$$\leqslant \sum_{j=1}^{k} \int_{[t_j, t_j+h_j]} | D(t_j, h_j, x) - u'(t_j)\, x(t_j)|\,dt$$

$$+ \sum_{j=1}^{k} \int_{[t_j, t_j+h_j]} | u'(t_j)\, x(t_j) - v(t)|\,dt$$

$$+ \left| \int_I \sum_{\substack{i=0 \\ i \notin A}}^{n} \frac{u(s_{i+1}) - u(s_i)}{s_{i+1} - s_i} y(s_i)\, \chi_{[s_i, s_{i+1}]}(t)\,dt \right| + \left| \int_{\bigcup_{i=0, i \notin A}^{n} [s_i, s_{i+1}]} v(t)\,dt \right|$$

$$< 2\epsilon + \left| \sum_{\substack{i=0 \\ i \notin A}}^{n} (u(s_{i+1}) - u(s_i))\, y(s_i) \right| + 2\epsilon < 7\epsilon.$$

Further, by (2.13), we have

$$\left| \int_I du(t)\, x(t) - \int_I du(t)\, y(t) \right| < \epsilon.$$

Hence

$$\left| \int_I du(t)\, x(t) - \int_I u'(t)\, x(t)\,dt \right|$$

$$\leqslant \left| \int_I du(t)\, x(t) - \int_I du(t)\, y(t) \right|$$

$$+ \left| \int_I du(t)\, y(t) - \int_I w(t)\,dt \right| + \left| \int_I w(t)\,dt - \int_I v(t)\,dt \right|$$

$$+ \left| \int_I v(t)\,dt - \int_I u'(t)\, x(t)\,dt \right| < 10\epsilon.$$

THEOREM 2.3. *Let* $x \in L(X)$, $u \in V^\infty(X^*)$, $\| u \| = 1$. *Then we have*

$$\int_I du(t)\, x(t) = \int_I | x(t)|\,dt \tag{2.20}$$

if and only if

$$u'(t)\,x(t) = |\,x(t)| \qquad a.e.\ in\ I. \tag{2.21}$$

Proof. If (2.21) holds, then we have (2.20) by Lemma 2.2.

Conversely, let (2.20) be satisfied. Then by Lusin's theorem, for every $n \in \mathbb{N}$ there is an open set R_n such that $\mu(R_n) < 1/n$ and such that x is continuous on $I \backslash R_n$. Since for every $n \in \mathbb{N}$, (2.20) implies

$$\int_{I \backslash R_n} du(t)\,x(t) = \int_{I \backslash R_n} |\,x(t)|\,dt,$$

we have (2.21) by Lemma 2.1.

Remark 2.4. Let $u \in V^\infty(X^*)$, $x \in L(X)$, $y(t) = \alpha(t) \cdot x(t)$ a.e. in I, where α is a real function on I. Let $u'(t)\,x(t)$ exist a.e. in I. Then we obviously have $u'(t)\,y(t) = \alpha(t) \cdot u'(t)\,x(t)$ a.e. in I.

3. Best Approximation in $L(X)$

The following theorem characterizes elements of best approximation in $L(X)$. The equivalence (i) $\Leftrightarrow$ (iii) is a generalization of a well-known theorem for real-valued summable functions given by James [8] and Kripke and Rivlin [10]. Our proof of the implication (iii) $\Rightarrow$ (i) is a modification of Singer's proof [12].

THEOREM 3.1. *Let E be a linear subspace of $L(X)$, $x \in L(X) \backslash \bar{E}$, $e_0 \in E$. Then the following conditions are equivalent*:

(i) $e_0 \in P_E(x)$

(ii) *There exists a $u \in V^\infty(X^*)$, $\|\,u\,\| = 1$ such that we have*

$$u'(t)(x(t) - e_0(t)) = |\,x(t) - e_0(t)| \qquad a.e.\ in\ I \tag{3.1}$$

and

$$\int_I du(t)\,e(t) = 0 \qquad for\ every\ e \in E. \tag{3.2}$$

(iii) *There exists a $u \in V^\infty(X^*)$, $\|\,u\,\| = 1$, such that we have* (3.1) *and*

$$\left|\int_{R(x-e_0)} du(t)\,e(t)\right| \leqslant \int_{Z(x-e_0)} |\,e(t)|\,dt \qquad for\ all\ e \in E. \tag{3.3}$$

Proof. (i) implies (ii): This is an immediate consequence of Singer's Theorem 1.1 [12], and Theorems 1.1 and 2.3 above.

(ii) implies (iii): For every $e \in E$ we have by (3.2)

$$\left| \int_{R(x-e_0)} du(t)\, e(t) \right| \leqslant \left| \int_{Z(x-e_0)} du(t)\, e(t) \right| \leqslant \int_{Z(x-e_0)} | e(t)|\, dt.$$

(iii) implies (i): Let $e \in E$. It follows from the Hahn–Banach theorem that there exists a $u_1 \in V^\infty(X^*)$, $\| u_1 \| = 1$, such that we have

$$\int_I du_1(t)(e_0(t) - e(t)) = \int_I | e_0(t) - e(t)|\, dt.$$

Hence

$$\int_{Z(x-e_0)} du_1(e_0 - e) = \int_{Z(x-e_0)} | e_0 - e |\, dt.$$

Defining

$$u_2(t) = -\frac{\int_{R(x-e_0)} du(e_0 - e)}{\int_{Z(x-e_0)} | e_0 - e |\, dt} \cdot u_1(t) \qquad \text{if } \int_{Z(x-e_0)} | e_0 - e |\, dt \neq 0,$$

$$= 0 \qquad \text{if } \int_{Z(x-e_0)} | e_0 - e |\, dt = 0,$$

we obtain $\| u_2 \| \leqslant 1$ and

$$\int_{R(x-e_0)} du(e_0 - e) + \int_{Z(x-e_0)} du_2(e_0 - e) = 0.$$

Hence

$$\begin{aligned} \| x - e_0 \| &= \int_{R(x-e_0)} du(x - e_0) = \int_{R(x-e_0)} du(x - e_0) + \int_{R(x-e_0)} du(e_0 - e) \\ &\quad + \int_{Z(x-e_0)} du_2(e_0 - e) + \int_{Z(x-e_0)} du_2(x - e_0) \\ &= \int_{R(x-e_0)} du(x - e) + \int_{Z(x-e_0)} du_2(x - e) \\ &\leqslant \int_{R(x-e_0)} | x - e |\, dt + \int_{Z(x-e_0)} | x - e |\, dt = \| x - e \|. \end{aligned}$$

The next theorem gives equivalent conditions for a linear subspace E of $L(X)$ to be a U-space. Similar theorems for real-valued summable functions have been proved by Cheney and Wulbert [5, Theorem 21] and Singer [12, Theorem 3.4].

LEMMA 3.2. *Let X be a strict convex Banach space and let $x \in L(X)$ have two best approximations $e_1, e_2 \in E$, $e_1 \neq e_2$. Then we have*

$$Z(x - e_0) \subset Z(e_1 - e_2), \tag{3.4}$$

where $e_0 = e_1/2 + e_2/2$ and there exists a real nonnegative function α such that

$$x(t) - e_1(t) = \alpha(t) \cdot (x(t) - e_2(t)) \qquad \textit{a.e. in } R(x - e_2). \tag{3.5}$$

Proof. Since $e_1, e_2, e_0 \in P_E(x)$, we have

$$\int_I (| x - e_1 | + | x - e_2 | - 2 | x - e_0 |) \, dt = 0.$$

The integrand being nonnegative, we must have

$$| x(t) - e_1(t)| + | x(t) - e_2(t)| = 2 | x(t) - e_0(t)| \qquad \text{a.e. in } I,$$

which implies (3.4) (this argument is due to Cheney and Wulbert [5]). Since X is strict convex, there exists a real nonnegative function α such that (3.5) holds.

THEOREM 3.3. *Let X be a strict convex Banach space, E a linear subspace of $L(X)$. Then the following conditions are equivalent:*

(i) *E is not a U-space.*

(ii) *There exist an $x \in E^0$, an $e_0 \in E \backslash \{0\}$, and a real function α, $| \alpha | \leqslant 1$, such that*

$$e_0(t) = \alpha(t) \cdot x(t) \qquad \textit{a.e. in } I. \tag{3.6}$$

(iii) *There exist a $u \in V^\infty(X^*)$, $\| u \| = 1$, and an $e_0 \in E \backslash \{0\}$ such that*

$$\int_I du(t) \, e(t) = 0 \qquad \textit{for every } e \in E \tag{3.7}$$

and

$$| u'(t) \, e_0(t)| = | e_0(t)| \qquad \textit{a.e. in } I. \tag{3.8}$$

Proof. (i) implies (ii): If E is not a U-space then there exist an $x \in E^0$ and an $e_0 \in E \backslash \{0\}$ such that e_0, $-e_0 \in P_E(x)$ (see e.g., [11, remark following 1.3]). By Lemma 3.2 there exist real nonnegative functions α_1 and α_2 such that we have

$$\begin{aligned} x(t) - e_0(t) &= \alpha_1(t) \cdot x(t) \qquad \text{a.e. in } R(x), \\ x(t) + e_0(t) &= \alpha_2(t) \cdot x(t) \qquad \text{a.e. in } R(x). \end{aligned}$$

Since α_1 and α_2 are nonnegative, we must have

$$|\alpha_2(t) - 1| \leqslant 1 \qquad \text{a.e. in } R(x).$$

On the other hand, we have by Lemma 3.2, $Z(x) \subset Z(e_0)$. Thus the function

$$\begin{aligned} \alpha(t) &= \alpha_2(t) - 1 \qquad && \text{for } t \in R(x) \\ &= 0 && \text{for } \ t \in Z(x) \end{aligned}$$

has the required properties.

(ii) implies (iii): If (ii) is satisfied then by Theorem 3.1 there is a $u \in V^\infty(X^*)$, $\| u \| = 1$, such that we have (3.7) and such that $u'(t)\, x(t) = | x(t)|$ a.e. in I. Hence by Remark 2.4 we obtain

$$| u'(t)\, e_0(t)| = | u'(t)(\alpha(t) \cdot x(t))| = | \alpha(t)| \cdot | x(t)| = | e_0(t)| \qquad \text{a.e. in } I.$$

(iii) implies (i): Let us define

$$y(t) = 2e_0(t) \cdot \operatorname{sign} u'(t)\, e_0(t) \qquad \text{for} \quad t \in I.$$

Then, by Remark 2.4, we obviously have $u'(t)\, y(t) = | y(t)|$ a.e. in I and $u'(t)(y(t) - e_0(t)) = | y(t) - e_0(t)|$ a.e. in I. Thus by Theorem 3.1, 0 and e_0 are best approximations of y.

Remark 3.4. The condition "There exist an $x \in E^0$ and an $e \in E \backslash \{0\}$ such that $Z(e) \supset Z(x)$," which in the case of real-valued functions is necessary and sufficient for E not to be a U-space [5], is not sufficient in the case of vector-valued functions.

EXAMPLE. Let $X = \mathbb{R}^2$ with the Euclidean norm $| \cdot |$. Then every $x \in L(X)$ has the form $x = (y, z)$, where $y, z \in L_1$, L_1 the space of all Lebesgue-summable real-valued functions. Let

$$E = \{e;\, e = (f, 0), f \in L_1\}.$$

Then we have for $x \in L(X)$, $x = (y, z)$, and $e \in E$, $e = (f, 0)$,

$$\| x - e \| = \int_I | x - e |\, dt = \int_I (| y - f |^2 + | z |^2)^{1/2}\, dt.$$

Thus $(y, 0)$ is the only best approximation of x in E and E is a U-space. On the other hand, for every $x = (0, z) \in E^0$ there exists an $e \in E \backslash \{0\}$, namely $e = (z, 0)$, such that $Z(e) \supset Z(x)$.

Remark 3.5. If X is not a strict convex Banach space, condition (ii) of Theorem 3.3 is not necessary for E not to be a U-space.

EXAMPLE. Let $X = \mathbb{R}^2$ with the norm $|x| = |(x_1, x_2)| = \text{Max}(|x_1|, |x_2|)$, E the same subspace of $L(X)$ as in Remark 3.4. Since for every $x \in L(X)$ of the form $x = (0, y)$ and $e \in E$, $e = (f, 0)$,

$$\| x - e \| = \int_I \text{Max}(|f|, |y|) \, dt,$$

every $e = (f, 0) \in E$ such that $|f| \leqslant |y|$ is a best approximation of x. On the other hand, if there exist an $x \in E^0$, $x = (y, z)$, an $e \in E$, $e = (f, 0)$, and a real function α such that $e = \alpha \cdot x$ a.e. in I, we must have $|y| \leqslant |z|$ a.e. in I which implies that $e = 0$ a.e. in $Z(z)$ and $\alpha = 0$ a.e. in $R(z)$. Thus $e = 0$ is the only element of E with the property $e = \alpha \cdot x$.

4. AN APPLICATION TO SIMULTANEOUS APPROXIMATION

Let $m, n \in \mathbb{N}$, $f_1, \ldots, f_m \in L_1$, and P_n be the space of all polynomials of degree less than or equal to n. Carroll and McLaughlin [4] considered the problem of finding a $p_0 \in P_n$ such that

$$\sum_{i=1}^{m} \int_I |f_i - p_0| \, dt = \inf_{p \in P_n} \sum_{i=1}^{m} \int_I |f_i - p| \, dt. \tag{4.1}$$

As remarked in [4], if m is even, the best approximation in this sense need not be unique.

Let $X = \mathbb{R}^m$ with the norm $|x| = |(x_1, \ldots, x_m)| = \sum_{i=1}^{m} |x_i|$ and let E be the space of all $e = (e_1, \ldots, e_m) \in L(X)$ such that there exists a $p \in P_n$ with $e_i = p$ for every $i = 1, \ldots, m$. Obviously, problem (4.1) is equivalent to the problem of finding for $x \in L(X)$ a best approximation in E. We show that there is a norm in X, namely every strict convex norm, such that the best approximation is always unique. We formulate this more generally.

Let A be an arbitrary set of indices, $X = \mathbb{R}^A$ a strict convex Banach space, Q a subspace of L_1 such that $\mu(Z(q)) = 0$ for every $q \in Q \backslash \{0\}$. For every $q \in Q$ let $L(X)$ contain the element $e = \{e_a\}$, $e_a = q$, for every $a \in A$. Let E be the subspace of all such elements. An element $x = \{x_a\} \in L(X)$ will be said to have the property (P) if there are indices a and b in A such that $x_a \neq x_b$ in a set of positive measure.

THEOREM 4.1. *For every $x \in L(X)$ with the property (P) the set $P_E(x)$ contains at most one element.*

Proof. If there is an $x \in L\{X\}$ such that e and $-e \in E \backslash \{0\}$ are best approximations of x, then by the proof of Theorem 3.3 there is a real function α

such that $e = \alpha \cdot x$ a.e. in I. Since $e(t) \neq 0$ a.e. in I, we have $\alpha(t) \neq 0$ a.e. in I. Thus $x(t) = 1/\alpha(t) \cdot e(t)$ a.e. in I, which implies that x cannot have the property (P).

The following corollaries are immediate consequences of Theorem 4.1.

COROLLARY 4.2. *Let m be an integer, $m \geqq 2$. Then for every $x = (x_1, \ldots, x_m)$, $x_i \in L_1$, $i = 1, \ldots, m$, satisfying the condition (P) there is at most one $q_0 \in Q$ such that*

$$\int_I \left(\sum_{i=1}^{m} |\, x_i(t) - q_0(t)|^2 \right)^{1/2} dt = \inf_{q \in Q} \int_I \left(\sum_{i=1}^{m} |\, x_i(t) - q(t)|^2 \right)^{1/2} dt.$$

COROLLARY 4.3. *For every $x = \{x_i\}_{i=1}^{\infty}$, $x_i \in L_1$, $i \in \mathbb{N}$, satisfying the condition (P) and such that $\sup_{i \in \mathbb{N}} \int_I |\, x_i(t)|\, dt < +\infty$ there exists at most one $q_0 \in Q$ such that*

$$\int_I \left(\sum_{i=1}^{\infty} \frac{1}{2^i} |\, x_i(t) - q_0(t)|^2 \right)^{1/2} dt = \inf_{q \in Q} \int_I \left(\sum_{i=1}^{\infty} \frac{1}{2^i} |\, x_i(t) - q(t)|^2 \right)^{1/2} dt.$$

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